

Phase transitions in BHT Massive Gravity

Mahdis Ghodrati^a, Ali Naseh^b

^a*Michigan Center for Theoretical Physics, Randall Laboratory of Physics
University of Michigan, Ann Arbor, MI 48109-1040, USA*

^b*School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5531, Tehran, Iran*

E-mails: ghodrati@umich.edu, naseh@ipm.ir

Abstract

We present the Hawking-Page phase diagrams in Bergshoeff-Hohm-Townsend (BHT) gravity for the phase transition between AdS_3 and BTZ black hole, warped AdS_3 and warped BTZ black hole in grand canonical and in non-local/quadratic ensembles, Lifshitz black hole and the new hairy black hole solutions. As we expected for all of them except for the quadratic ensemble, the phase diagram is symmetric for the non-chiral theory of BHT. We also examine the laws of inner horizon mechanics for the warped AdS_3 black holes and proved that they are satisfied. Finally we briefly discuss the entanglement entropy of an interval in the warped CFT_2 which holographically is dual to the vacuum time-like warped AdS_3 or Gödel geometry.

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1 Introduction

In order to study the quantum field theories with momentum dissipation holographically, the holographic massive gravity theories (HMGs) have been exploited. Using these models one can study different field theory features such as DC resistivity, relaxation rates, or the effect of dissipations or disorders on the confinement-deconfinement phase transitions in strongly correlated systems [1].

There are different massive gravity models with multiple geometrical solutions and their corresponding field theory duals. One of these theories is the “Topological Massive Gravity” (TMG), which is the Einstein action, plus a Chern-Simons term breaking the parity. Recently in [2], the Hawking-Page phase transitions between the AdS_3 and BTZ solutions, and warped AdS_3 and warped BTZ black hole of TMG were investigated and the Gibbs free energies, local and global stability regions and the phase diagrams were presented.

There is yet another rich theory, the parity preserving Bergshoeff-Hohm-Townsend (BHT) or the “New Massive Gravity” (NMG), which has many different solutions as well, in addition to the thermal warped AdS_3 and warped BTZ black hole. The aim of this paper is, similar to [2], we study the Hawking-Page phase transitions between different solutions of NMG and therefore learning more about the properties of the dual CFTs. Particularly we study the phase transitions between the thermal AdS and BTZ black holes, the warped AdS and warped BTZ black holes in two different ensembles, the Lifshitz black hole and the new hairy black hole and their corresponding vacua.

The other motivation is to extend the AdS/CFT duality to other more general geometries. One would think that for doing so, the most direct way is to perturbatively deform the AdS_3 manifold to a warped AdS_3 geometry, [3] [4] [5], and then study the dual field theory. The initial works on this extension were done in [6], where the authors studied the magnetic deformation of S^3 and the electric/magnetic deformations of AdS_3 which still could remain a solution of string theory vacua. Then in [7] [8] [9], the dual field theories have been studied. In [9] the dual of warped AdS_3 were suggested to be the IR limit of the non-local deformed 2D D-brane gauge theory or the dipole CFT. Constructing this duality could lead to more information about the properties of these new field theories and also some properties of the dual bulk geometries, for instance the nature of the closed time-like curves (CTCs).

The Bergshoeff-Hohm-Townsend gravity have both warped AdS and warped BTZ black hole solutions. The deformed AdS_3 preserve a $SL(2, R) \times U(1)$ subgroup of $SL(2, R) \times SL(2, R)$ isometries. The obtained space-times called null, time-like or space-like warped AdS_3 (WAdS_3) corresponding to the norm of $U(1)$ killing vectors, where the time-like WAdS_3 is just the *Gödel* spacetime [3] [10].

Other extension of AdS/CFT includes AdS/CMT (Condensed Matter), AdS/QCD, dS/CFT, flat space holography, Kerr/CFT, etc. However, the dual CFT of these theories are not completely known. The advantages of WCFTs are that they possess many properties of CFTs and they can be derived from string theory and low-dimensional gravity theories and hence for studying them the known CFT techniques could be deployed.

The specific properties of this new class of WCFTs were studied in [11] and their entanglement entropies were first studied in [12] holographically and in a more recent work in [13] by using the Rindler method of WCFT. To further study this WAdS/WCFT duality, one could study other properties such as the instabilities of the solutions and the Hawking-Page phase transitions [14]. As the phase transitions from the thermal AdS or WAdS , to BTZ or warped BTZ black hole is dual to confining/deconfining phase transitions in the dual field theory, these models could be used in QCD or condensed matter systems with dissipations.

The plan of this paper is as follows. First we review two methods of finding the conserved charges for any solution of NMG, the ADT formalism and the $SL(2, R)$ reduction method. Mainly we use the general formulas from $SL(2, R)$ reduction method to calculate the conserved charges for any solution of NMG in different ensembles. Then by finding the free energies, we discuss the phase transitions between the vacuum AdS_3 and BTZ black hole solutions in section 2. We discuss the thermodynamics and local and global stability regions. In section 5 we calculate the free energies of warped AdS_3 vacuum and warped BTZ black hole solutions. We calculate the free energy of the WAdS_3 by three different methods and by doing so we could find a factor in the modular parameter which extends the result of [15] for calculating the free energy of WAdS_3 solutions in NMG. Then we present the phase diagrams of these solutions. In

section 7 we discuss the free energy and phase transitions of the Lifshitz and the new hairy black hole solution in NMG. We also discuss the inner horizon thermodynamics in section 8. In section 9, we discuss the entanglement entropy of the vacuum solutions corresponding to the WCFT dual of WAdS in NMG and then we conclude with a discussion in section 10.

2 The Bergshoeff-Hohm-Townsend Theory

The Bergshoeff-Hohm-Townsend (BHT) or the new massive gravity (NMG) is a higher-curvature extension of the Einstein-Hilbert action in three dimensions which is diffeomorphism and parity invariant. In the linearized level, it is equivalent to the unitary Pauli-Fierz action for a massive spin-2 field [16].

The action of NMG is

$$S = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g} \left[R - 2\Lambda + \frac{1}{m^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2 \right) \right], \quad (2.1)$$

where m is the mass parameter, Λ is a cosmological parameter and G_N is a three-dimensional Newton constant. In the case of $m \rightarrow \infty$, the theory reduces to the Einstein gravity and in the limit of $m \rightarrow 0$, it is just a pure fourth-order gravity.

The equation of motion from the action would be derived as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{m^2} K_{\mu\nu} = 0, \quad (2.2)$$

with the explicit form of $K_{\mu\nu}$ as in [16],

$$K_{\mu\nu} = \nabla^2 R_{\mu\nu} - \frac{1}{4} (\nabla_\mu \nabla_\nu R + g_{\mu\nu} \nabla^2 R) - 4R_\mu^\sigma R_{\sigma\nu} + \frac{9}{4} R R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(3R^{\alpha\beta} R_{\alpha\beta} - \frac{13}{8} R^2 \right). \quad (2.3)$$

The boundary terms of NMG which make the variational principle well-defined would be

$$S_{\text{Boundary}} = \frac{1}{16\pi G} \int_\sigma d^3x \sqrt{-g} \left(f^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) - \frac{1}{4} m^2 (f_{\mu\nu} f^{\mu\nu} - f^2) \right), \quad (2.4)$$

where $f_{\mu\nu}$, the rank two symmetric tensor is

$$f_{\mu\nu} = \frac{2}{m^2} (R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}). \quad (2.5)$$

This theory admits different solutions, such as the vacuum AdS₃, warped AdS₃, BTZ black hole, asymptotic warped AdS black hole, Lifshitz, Schrödinger and so on [16], [17]. We construct the phase diagrams between several of these solutions by comparing the *on-shell* free energies.

By constructing the *off-shell* free energies, one could even find all the states connecting any two solutions and therefore create a picture of continuous evolutions of the phase transitions; similar to the work in [18], who studied the continuous phase transition between the BTZ black hole with $M \geq 0$ and the thermal AdS soliton with $M = -1$ in the new massive gravity.

In the next section we review how one can calculate the conserved charges and we bring several general formulas for the solutions of NMG which could be used to find the on-shell Gibbs free energies. In section 4 we study the vacuum AdS₃ and BTZ solutions of NMG, the free energies and the phase diagrams. Then in section 5 we discuss the warped solutions and in section 7 we study the new hairy black hole solution of this theory.

3 Review of calculating conserved charges in BHT

In three dimensions, the conserved charges associated to a Killing vector ξ would be

$$\delta Q_\xi[\delta g, g] = \frac{1}{16\pi G} \int_0^{2\pi} \sqrt{-g} \epsilon_{\mu\nu\varphi} k_\xi^{\mu\nu} [\delta g, g] d\varphi. \quad (3.1)$$

As calculated in [19] for BHT, the ADT formalism would result in

$$k_\xi^{\mu\nu} = Q_R^{\mu\nu} + \frac{1}{2m^2} Q_K^{\mu\nu}, \quad (3.2)$$

where

$$Q_K^{\mu\nu} = Q_{R_2}^{\mu\nu} - \frac{3}{8} Q_{R^2}^{\mu\nu}, \quad (3.3)$$

and the term for each charge is

$$Q_R^{\mu\nu} \equiv \xi_\alpha \nabla^{[\mu} h^{\nu]\alpha} - \xi^{[\mu} \nabla_\alpha h^{\nu]\alpha} - h^{\alpha[\mu} \nabla_\alpha \xi^{\nu]} + \xi^{[\mu} \nabla^{\nu]} h + \frac{1}{2} h \nabla^{[\mu} \xi^{\nu]}, \quad (3.4)$$

$$Q_{R^2}^{\mu\nu} = 2R Q_R^{\mu\nu} + 4\xi^{[\mu} \nabla^{\nu]} \delta R + 2\delta R \nabla^{[\mu} \xi^{\nu]} - 2\xi^{[\mu} h^{\nu]\alpha} \nabla_\alpha R, \quad (3.5)$$

where

$$\delta R \equiv -R^{\alpha\beta} h_{\alpha\beta} + \nabla^\alpha \nabla^\beta h_{\alpha\beta} - \nabla^2 h, \quad (3.6)$$

and

$$\begin{aligned} Q_{R_2}^{\mu\nu} = & \nabla^2 Q_R^{\mu\nu} + \frac{1}{2} Q_{R^2}^{\mu\nu} - 2Q_R^{\alpha[\mu} R^{\nu]}_\alpha - 2\nabla^\alpha \xi^\beta \nabla_\alpha \nabla^{[\mu} h^{\nu]\beta} - 4\xi^\alpha R_{\alpha\beta} \nabla^{[\mu} h^{\nu]\beta} - R h^{[\mu}_\alpha \nabla^{\nu]} \xi^\alpha \\ & + 2\xi^{[\mu} R^{\nu]}_\alpha \nabla_\beta h^{\alpha\beta} + 2\xi_\alpha R^{\alpha[\mu} \nabla_\beta h^{\nu]\beta} + 2\xi^\alpha h^{\beta[\mu} \nabla_\beta R^{\nu]}_\alpha + 2h^{\alpha\beta} \xi^{[\mu} \nabla_\alpha R^{\nu]}_\beta \\ & - (\delta R + 2R^{\alpha\beta} h_{\alpha\beta}) \nabla^{[\mu} \xi^{\nu]} - 3\xi^\alpha R^{\alpha[\mu} \nabla^{\nu]} h - \xi^{[\mu} R^{\nu]\alpha} \nabla_\alpha h. \end{aligned} \quad (3.7)$$

For the three dimensional case of (t, r, ϕ) , the mass and angular momentum in three dimensions would be [19]

$$M = \frac{1}{4G} \sqrt{-\det g} Q^{rt}(\xi_T) \Big|_{r \rightarrow \infty}, \quad J = \frac{1}{4G} \sqrt{-\det g} Q^{rt}(\xi_R) \Big|_{r \rightarrow \infty}, \quad (3.8)$$

where

$$\xi_T = \frac{1}{L} \frac{\partial}{\partial t}, \quad \xi_R = \frac{\partial}{\partial \phi}. \quad (3.9)$$

3.1 The $SL(2, R)$ reduction method

One can also derive the charges by $SL(2, R)$ reduction method which changes the metric to $SO(1, 2)$ from. For doing so one should write the metric in the form of [19]

$$ds^2 = \lambda_{ab}(\rho) dx^a dx^b + \frac{d\rho^2}{\zeta^2 U^2(\rho)}, \quad x^a = (t, \phi). \quad (3.10)$$

Since there is a reparametrization invariance with respect to the radial coordinate, one can write the function U such that $\det \lambda = -U^2$ and this would give $\sqrt{-g} = 1/\zeta$. One then applies the equations of motions (EOMs) and the Hamiltonian constraint and then by integrating the EOM one can derive the “*super angular momentum*” vector.

So first one parameterize the matrix λ as

$$\lambda_{ab} = \begin{pmatrix} X^0 + X^1 & X^2 \\ X^2 & X^0 - X^1 \end{pmatrix}, \quad (3.11)$$

where $\mathbf{X} = (X^0, X^1, X^2)$ would be the $SO(1, 2)$ vector.

Then one applies the reduced equation of motion and the Hamiltonian constraint as [20]

$$\begin{aligned} & \mathbf{X} \wedge (\mathbf{X} \wedge \mathbf{X}''''') + \frac{5}{2} \mathbf{X} \wedge (\mathbf{X}' \wedge \mathbf{X}''') + \frac{3}{2} \mathbf{X}' \wedge (\mathbf{X} \wedge \mathbf{X}''''') + \frac{9}{4} \mathbf{X}' \wedge (\mathbf{X}' \wedge \mathbf{X}'') \\ & - \frac{1}{2} \mathbf{X}'' \wedge (\mathbf{X} \wedge \mathbf{X}'') - \left[\frac{1}{8} (\mathbf{X}'^2) + \frac{m^2}{\zeta^2} \right] \mathbf{X}'' = 0, \end{aligned} \quad (3.12)$$

$$\begin{aligned} H & \equiv (\mathbf{X} \wedge \mathbf{X}') \cdot (\mathbf{X} \wedge \mathbf{X}''''') - \frac{1}{2} (\mathbf{X} \wedge \mathbf{X}'')^2 + \frac{3}{2} (\mathbf{X} \wedge \mathbf{X}') \cdot (\mathbf{X}' \wedge \mathbf{X}'') \\ & + \frac{1}{32} (\mathbf{X}'^2)^2 + \frac{m^2}{2\zeta^2} (\mathbf{X}'^2) + \frac{2m^2 \Lambda}{\zeta^4} = 0. \end{aligned} \quad (3.13)$$

From these two equations one can find ζ^2 and Λ . Then one can define the vector

$$\mathbf{L} \equiv \mathbf{X} \wedge \mathbf{X}', \quad (3.14)$$

where $' \equiv \frac{d}{d\rho}$.

Finally the *super angular momentum* of NMG $\mathbf{J} = (J^0, J^1, J^2)$ would be

$$\mathbf{J} = \mathbf{L} + \frac{\zeta^2}{m^2} \left[2\mathbf{L} \wedge \mathbf{L}' + \mathbf{X}^2 \mathbf{L}'' + \frac{1}{8} (\mathbf{X}'^2 - 4\mathbf{X} \cdot \mathbf{X}'') \mathbf{L} \right], \quad (3.15)$$

where the products are defined as

$$\mathbf{A} \cdot \mathbf{B} = \eta_{ij} A^i B^j, \quad (\mathbf{A} \wedge \mathbf{B})^i = \eta^{im} \epsilon_{mjk} A^j B^k, \quad (\epsilon_{012} = 1). \quad (3.16)$$

That being so for the case of NMG one would have [19]

$$\begin{aligned} \eta \left[\sigma Q_R^{\rho t} + \frac{1}{m^2} Q_K^{\rho t} \right]_{\zeta_T} &= \frac{1}{L} \left[-\frac{\zeta^2}{2} \delta J^2 + \Delta_{Cor} \right], \\ \eta \left[\sigma Q_R^{\rho t} + \frac{1}{m^2} Q_K^{\rho t} \right]_{\zeta_R} &= \frac{\zeta^2}{2} \delta(J^0 - J^1), \end{aligned} \quad (3.17)$$

where η and σ are ± 1 , depending on the sign in the action. Based on eq. 2.1, both of η and σ would be positive in our case. Also Δ_{Cor} , the correction term to the mass, for the NMG would be

$$\Delta_{Cor} = \Delta_R + \Delta_K, \quad (3.18)$$

where

$$\begin{aligned} \Delta_R &= \frac{\zeta^2}{2} [-(\mathbf{X} \cdot \delta \mathbf{X}')], \\ \Delta_K &= \frac{\zeta^4}{m^2} \left[-U^2(\mathbf{X}'' \cdot \delta \mathbf{X}') + \frac{U^2}{2} [(U\delta U)''' - (\mathbf{X} \cdot \delta \mathbf{X}')'' - \frac{1}{2}(\mathbf{X}' \cdot \delta \mathbf{X}')'] \right. \\ &\quad - \frac{UU'}{4} [(U\delta U)'' - \frac{5}{2}(\mathbf{X}' \cdot \delta \mathbf{X}')] + [\mathbf{X}'^2 - (UU')'](U\delta U)' + UU'(\mathbf{X}'' \cdot \delta \mathbf{X}) \\ &\quad \left. + \left[\frac{5}{4}(UU')' - \frac{21}{16}\mathbf{X}'^2 \right](\mathbf{X} \cdot \delta \mathbf{X}') + \left[-\frac{1}{2}(UU')'' + \frac{9}{4}(\mathbf{X}' \cdot \mathbf{X}'') \right] U\delta U \right]. \end{aligned} \quad (3.19)$$

Then the mass and angular momentum in NMG would be

$$\begin{aligned} \mathbf{M} &= \frac{1}{4G} \sqrt{-\det g} \left[Q_R^{rt} + \frac{1}{m^2} Q_K^{rt} \right]_{\zeta_T, r \rightarrow \infty}, \\ \mathbf{J} &= \frac{1}{4G} \sqrt{-\det g} \left[Q_R^{rt} + \frac{1}{m^2} Q_K^{rt} \right]_{\zeta_R, r \rightarrow \infty}. \end{aligned} \quad (3.20)$$

Also for calculating entropy for any solution of NMG, we can use the following relation from [20]

$$\mathbf{S} = \frac{A_h}{4G} \left(1 + \frac{\zeta^2}{2m^2} \left[(\mathbf{X} \cdot \mathbf{X}'') - \frac{1}{4}(\mathbf{X}'^2) \right] \right). \quad (3.21)$$

Now using these relations one can derive the charges, Gibbs free energies and the phase diagrams of several solutions of NMG.

3.2 Examples of conserved charges of BHT solutions

First for the warped AdS black hole in the “grand canonical ensemble” [2],

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r^2}{l^2} - \frac{H^2(-r^2-4lJ+8l^2M)^2}{4l^3(lM-J)} + 8M & 0 & 4J - \frac{H^2(4lJ-r^2)(-r^2-4lJ+8l^2M)}{4l^2(lM-J)} \\ 0 & \frac{1}{\frac{16J^2}{r^2} + \frac{r^2}{l^2} - 8M} & 0 \\ 4J - \frac{H^2(4lJ-r^2)(-r^2-4lJ+8l^2M)}{4l^2(lM-J)} & 0 & r^2 - \frac{H^2(4Jl-r^2)^2}{4l(lM-J)} \end{pmatrix}, \quad (3.22)$$

by reparametrizing the radial coordinate as $r^2 \rightarrow \rho$, and then by applying the equation of motion and hamiltonian constraints 3.12 and 3.13 one can find

$$\zeta^2 = \frac{8l^2m^2}{(1-2H^2)(17-42H^2)}, \quad \Lambda = \frac{m^2(84H^4+60H^2-35)}{(17-42H^2)^2}. \quad (3.23)$$

From the above relation one can see that the acceptable region for Λ is

$$\frac{-35m^2}{289} < \Lambda < \frac{m^2}{21}, \quad (3.24)$$

and the special case of $\Lambda = \frac{m^2}{21}$ corresponds to the $\text{AdS}_2 \times S^1$.

Now for the metric 6.1, the components of the super angular momentum would be

$$J^0 = -\frac{H^2(1+l^2)}{4l^3(-J+lM)}, \quad J^1 = \frac{H^2(-1+l^2)}{4l^3(-J+lM)}, \quad J^2 = \frac{H^2}{2l^2(J-lM)}. \quad (3.25)$$

Then using 3.26 and 3.20 one can find the charges as

$$\mathbf{M} = \frac{16(1-2H^2)^{3/2}M}{GL(17-42H^2)}, \quad \mathbf{J} = \frac{16(1-2H^2)^{3/2}J}{G(17-42H^2)}. \quad (3.26)$$

One should note that Δ_{Cor} would be zero here.

For the above metric using 3.21, the entropy would be

$$\mathbf{S} = \frac{16\pi(1-2H^2)^{3/2}}{G(17-42H^2)} \sqrt{l^2M + \sqrt{l^4M^2 - J^2l^2}}. \quad (3.27)$$

We can then study this black hole solution in another ensemble. The asymptotically warped AdS_3 black hole in NMG in the ADM form and therefore in the “*quadratic/non-local ensemble*” would be in the following form

$$\begin{aligned} \frac{ds^2}{l^2} = & dt^2 + \frac{dr^2}{(\nu^2+3)(r-r_+)(r-r_-)} + (2\nu r - \sqrt{r_+r_-(\nu^2+3)})dtd\varphi \\ & + \frac{r}{4} \left[3(\nu^2-1)r + (\nu^2+3)(r_++r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)} \right] d\varphi^2. \end{aligned} \quad (3.28)$$

So using 3.12 and 3.13, one would have

$$\zeta^2 = \frac{8m^2}{l^4(20\nu^2 - 3)}, \quad \Lambda = \frac{m^2(9 - 48\nu^2 + 4\nu^4)}{(3 - 20\nu^2)^2}. \quad (3.29)$$

The components of the super angular momentum would be

$$\begin{aligned} J^0 &= -\frac{l^4\nu(\nu^2 + 3) \left(4 - 2r_-\nu\sqrt{r_+r_-(\nu^2 + 3)} - 2r_+\nu\sqrt{r_+r_-(\nu^2 + 3)} + r_+r_- (5\nu^2 + 3) \right)}{2(20\nu^2 - 3)}, \\ J^1 &= \frac{l^4\nu(\nu^2 + 3) \left(-4 - 2r_-\nu\sqrt{r_+r_-(\nu^2 + 3)} - 2r_+\nu\sqrt{r_+r_-(\nu^2 + 3)} + r_+r_- (5\nu^2 + 3) \right)}{2(20\nu^2 - 3)}, \\ J^2 &= -\frac{2l^4\nu(\nu^2 + 3) \left((r_+ + r_-)\nu - \sqrt{r_+r_-(\nu^2 + 3)} \right)}{20\nu^2 - 3}. \end{aligned}$$

Then by using 3.26 and 3.20 one could find the conserved charges [20] [21]

$$\begin{aligned} \mathbf{M} &= \frac{\nu(\nu^2 + 3)}{2G(20\nu^2 - 3)} \left((r_+ + r_-)\nu - \sqrt{r_+r_-(\nu^2 + 3)} \right), \\ \mathbf{J} &= \frac{\nu(\nu^2 + 3)}{4Gl(20\nu^2 - 3)} \left((5\nu^2 + 3)r_+r_- - 2\nu\sqrt{r_+r_-(3 + \nu^2)}(r_+ + r_-) \right), \\ \mathbf{S} &= \frac{4\pi l\nu^2}{G(20\nu^2 - 3)} \sqrt{r_+r_-(\nu^2 + 3) + 4r_+\nu \left(r_+\nu - \sqrt{r_+r_-(\nu^2 + 3)} \right)}. \end{aligned} \quad (3.30)$$

As another example of a practical solution of NMG in condensed matter, one could also study the conserved charges of the Lifshitz geometry

$$ds^2 = -\frac{r^{2z}}{l^{2z}}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}d\vec{x}^2. \quad (3.31)$$

Here $\zeta^2 = -\frac{2m^2l^{2+2z}}{1+z(z-3)}$ and the vector of super angular momentum would be zero. The case of $z = 3$ and $z = \frac{1}{3}$ could be a solution of the simple NMG with no matter content. For the case of $z = 3$, one would have $\zeta^2 = -2l^8m^2$.

Now considering the Lifshitz black hole solutions [22]

$$ds^2 = -\frac{r^{2z}}{l^{2z}} \left[1 - M \left(\frac{l}{r} \right)^{\frac{z+1}{2}} \right] dt^2 + \frac{l^2}{r^2} \left[1 - M \left(\frac{l}{r} \right)^{\frac{z+1}{2}} \right]^{-1} dr^2 + \frac{r^2}{l^2} d\varphi^2, \quad (3.32)$$

by taking $r \rightarrow \left(\rho(z+1) \right)^{\frac{1}{1+z}}$, one would have $\sqrt{-g} = 1/\zeta = l^{-z}$ which would result in

$$\mathbf{M} = -\frac{\pi M^2(z+1)^2(3z-5)}{16\kappa(z-1)(z^2-3z+1)}, \quad \mathbf{J} = 0, \quad (3.33)$$

in accordance with [22]. This would lead us to the following Gibbs free energy

$$G_{\text{Lifshitz BH}} = \frac{M^2 \pi z (z+1)^2 (3z-5)}{16k(z-1)(z(z-3)+1)}. \quad (3.34)$$

Comparing this result with the free energy of the Lifshitz metric, one can see that in NMG always the Lifshitz black hole would be the dominant phase.

4 Phase transitions of AdS₃ solution

The vacuum AdS₃ solution is

$$ds_{\text{AdS}_3}^2 = l^2 (d\rho^2 - \cosh^2 \rho \, dt^2 + \sinh^2 \rho \, d\phi^2), \quad (4.1)$$

where [23]

$$1/l^2 = 2m^2 (1 \pm \sqrt{1 + \frac{\Lambda}{m^2}}), \quad (4.2)$$

and the boundary where the dual CFT is defined is located at $\rho \rightarrow \infty$.

For this case, we use the relation $G(T, \Omega) = TS[g_c]$ to find the Gibbs free energy, where g_c is the Euclidean saddle and $\tau = \frac{1}{2\pi}(-\beta\Omega_E + i\frac{\beta}{l})$ is the modular parameter. We work in the regimes that the saddle-point approximation could be used.

First we need to find the free energy of the vacuum solution. In [15] [24], the authors derived a general result for deriving the action of the thermal AdS₃ in any theory as,

$$S_E(\text{AdS}(\tau, \tilde{\tau})) = \frac{i\pi}{12l}(c\tau - \tilde{c}\tilde{\tau}). \quad (4.3)$$

Also the modular transformed version of this equation would give the thermal action of the BTZ black hole. By changing the boundary torus as $\tau \rightarrow -\frac{1}{\tau}$, and then by using the modular invariance, one would have

$$ds_{\text{BTZ}}^2 \left[-\frac{1}{\tau} \right] = ds_{\text{AdS}}^2[\tau], \quad (4.4)$$

so

$$S_E(\text{BTZ}(\tau, \tilde{\tau})) = \frac{i\pi}{12l}(\frac{c}{\tau} - \frac{\tilde{c}}{\tilde{\tau}}). \quad (4.5)$$

In this equation the contributions of the quantum fluctuations of the massless field is neglected as they are suppressed for large β .

One should notice that this equation and its modular transformed version are only true for the AdS₃ and not particularly for the “warped AdS₃” or “asymptotically warped AdS black holes”. This equation is correct as in the Lorentzian signature, the thermal AdS₃ has the same form as in the global coordinates and also the global AdS₃ corresponds to NS-NS vacuum with zero Virasoro modes [15]. These statements are not particularly correct for geometries with other asymptotics than AdS, specifically geometries such

as warped AdS₃.

In the next section, by redefinition of the modular parameter τ , and deriving the free energy by three different methods, we find a new equation for the thermal action of the warped AdS₃ in NMG case as well.

So for now, inserting the central charges of the NMG [25] [26],

$$c_L = c_R = \frac{3l}{2G_N} \left(1 - \frac{1}{2m^2 l^2}\right), \quad (4.6)$$

and the modular parameter $\tau = \frac{1}{2\pi}(-\beta\Omega_E + i\frac{\beta}{l})$ in Eq 4.3 would result in

$$S_E = -\frac{1}{8lTG_N} \left(1 - \frac{1}{2m^2 l^2}\right). \quad (4.7)$$

This relation, unlike the corresponding equation in the TMG case, does not depend on the angular velocity Ω_E . This is because the NMG has chiral symmetry and so the central charges are equal which causes that the terms containing Ω cancel out.

So the Gibbs free energy would be

$$G_{AdS}(T, \Omega) = -\frac{1}{8lG_N} \left(1 - \frac{1}{2m^2 l^2}\right). \quad (4.8)$$

Just by considering this equation, one can see that the stability condition of the vacuum AdS₃ in NMG would be $m^2 l^2 > \frac{1}{2}$ which is different from the Einstein theory.

Additionally, the NMG theory also admits a general BTZ solution. The rotating BTZ black hole metric solution in this theory would be of the following form

$$ds^2 = \left(-\frac{2\rho}{\tilde{l}^2} + \frac{M}{2}\right)dt^2 - jdt d\phi + \left(2\rho + \frac{M\tilde{l}^2}{2}\right)d\phi^2 + \frac{d\rho^2}{\left(\frac{4\rho^2}{\tilde{l}^2} - \frac{M^2\tilde{l}^2 - j^2}{4}\right)}, \quad (4.9)$$

where the AdS curvature is again [20], $l^{-2} = 2m^2[1 \pm \sqrt{1 + \frac{\Lambda}{m^2}}]$, and M is the ADM mass of the black hole. If we aim to write the metric in the ADM form,

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)^2} + R(r)^2 (N^\phi(r) dt + d\phi)^2, \quad (4.10)$$

we need to go from the coordinate system (t, ρ, ϕ) to (t, r, ϕ) , so we should change the radial coordinate as $\rho = r^2/2 - M\tilde{l}^2/4$, and then re-parametrize the three coordinates as $r \rightarrow \tilde{l}r$, $t \rightarrow -lt$ and $\phi \rightarrow L\phi/\tilde{l}$.

Then the metric becomes [27]

$$ds^2 = l^2 \left[-\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} dt \right)^2 \right]. \quad (4.11)$$

The Hawking temperature of this black hole would be [27]

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi l} \frac{\partial_r N}{\sqrt{g_{rr}}} \Big|_{r=r_+} = \frac{r_+}{2\pi l} \left(1 - \frac{r_-^2}{r_+^2}\right), \quad (4.12)$$

the entropy is

$$S_{BH} = \frac{\pi^2 l}{3} c(T_L + T_R), \quad (4.13)$$

and the angular velocity at the horizon is defined as [27]

$$\Omega_H = \frac{1}{l} N^\phi(r_+) = \frac{1}{l} \frac{r_-}{r_+}. \quad (4.14)$$

Also the left and right temperatures are given by [28]

$$T_L = \frac{r_+ + r_-}{2\pi l} = \frac{T}{1 - l\Omega}, \quad T_R = \frac{r_+ - r_-}{2\pi l} = \frac{T}{1 + l\Omega}, \quad (4.15)$$

and the left and right energies can be defined as follows

$$E_L \equiv \frac{\pi^2 l}{6} c_L T_L^2, \quad E_R \equiv \frac{\pi^2 l}{6} c_R T_R^2. \quad (4.16)$$

These parameters are related to the mass and angular momentum as [27]

$$M = E_L + E_R, \quad J = l(E_L - E_R). \quad (4.17)$$

The horizons of the BTZ black hole are located at

$$r_+ = \sqrt{2\left(\frac{M\tilde{l}^2}{4} + \frac{\tilde{l}}{4}\sqrt{M^2\tilde{l}^2 - j^2}\right)} = \frac{2\pi l T}{1 - \Omega^2 l^2}, \quad r_- = \sqrt{2\left(\frac{M\tilde{l}^2}{4} - \frac{\tilde{l}}{4}\sqrt{M^2\tilde{l}^2 - j^2}\right)} = \frac{2\pi \Omega l^2 T}{1 - \Omega^2 l^2}. \quad (4.18)$$

For the BTZ black hole in NMG which has an asymptotic AdS geometry, again the central charges would be

$$c_L = c_R = \frac{3l}{2G_N} \left(1 - \frac{1}{2m^2 l^2}\right). \quad (4.19)$$

For having a physical theory, the central charge and the mass of the BTZ black hole should be positive which again sets the condition of $m^2 l^2 > \frac{1}{2}$.

These parameters would satisfy the first law of thermodynamics,

$$dM = T_H dS_{BH} + \Omega_H dJ, \quad (4.20)$$

and the integral of it would satisfy the Smarr relation [27],

$$M = \frac{1}{2} T_H S_{BH} + \Omega_H J. \quad (4.21)$$

Now one can read the Gibbs free energy from the following relation,

$$G = M - T_H S_{BH} - \Omega_H J. \quad (4.22)$$

So using all the above equations, the Gibbs free energy of the BTZ in NMG would be

$$G_{BTZ}(T, \Omega) = -\frac{\pi^2 T^2 (2m^2 l^2 - 1)}{4G_N m^2 l (1 - l^2 \Omega^2)}. \quad (4.23)$$

This result can also be rederived by considering the modular invariance. Therefore using the relation 4.5 and $G(T, \Omega) = TS[g_c]$ again denotes the applicability of 4.3 for the AdS_3 case in NMG. From this relations one can see that for small rotations Ω as also explained in [29], the thermal stability condition for BTZ black hole in NMG is $m^2 l^2 > \frac{1}{2}$, regardless of the size of the event-horizon. For this case, the Hawking-Page phase transition can occur between the BTZ black hole and the thermal solution, while for the case of $m^2 l^2 < \frac{1}{2}$ the fourth-order curvature terms is dominant and in this case an inverse Hawking-Page phase transition between the BTZ black hole and the ground state massless BTZ black hole can occur [29].

One can also discuss the interpolations and the continuous phase transitions between these phases such as the scenario in [18].

We now extend these results to the higher angular momentums.

4.1 The stability conditions

For checking the local stability we find the Hessain, H , of the free energy $G(T, \Omega)$ of the BTZ metric as

$$H = \begin{pmatrix} \frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Omega} \\ \frac{\partial^2 G}{\partial \Omega \partial T} & \frac{\partial^2 G}{\partial \Omega^2} \end{pmatrix} = \begin{pmatrix} \frac{\pi^2 (2m^2 l^2 - 1)}{2G_N m^2 (\Omega^2 l^2 - 1)} & \frac{\pi^2 l^2 (1 - 2l^2 m^2) T \Omega}{G_N m^2 (\Omega^2 l^2 - 1)^2} \\ \frac{\pi^2 l^2 (1 - 2m^2 l^2) T \Omega}{G_N m^2 (\Omega^2 l^2 - 1)^2} & \frac{\pi^2 T^2 (2m^2 l^2 - 1) (l^2 + 3l^4 \Omega^2)}{2G_N m^2 (\Omega^2 l^2 - 1)^3} \end{pmatrix}. \quad (4.24)$$

In the region where both of its eigenvalues are negative, the system is stable. By finding the eigenvalues of the above matrix and then by assuming $G_N = l = 1$, the stable region would be found as $m^2 > 1$ and $\Omega^2 < 1$ for any T , similar to the stability region of TMG in [2].

Now for calculating the global stability, we calculate the difference of the free energies of AdS and BTZ case as

$$\Delta G = G_{AdS} - G_{BTZ} = \frac{2m^2 l^2 - 1}{4G_N m^2 l} \left(\frac{\pi^2 T^2}{1 - l^2 \Omega^2} - \frac{1}{4l^2} \right). \quad (4.25)$$

4.2 Phase diagrams

When $\Delta G > 0$, the BTZ black hole is the dominant phase and when $\Delta G < 0$, the thermal AdS_3 is dominant. Now if we assume $G_N = l = 1$ then we can show the phase diagrams as in Figures 1 and 2. One can notice that, since in NMG unlike the TMG case, the parity is conserved, the phase diagrams would be symmetric.

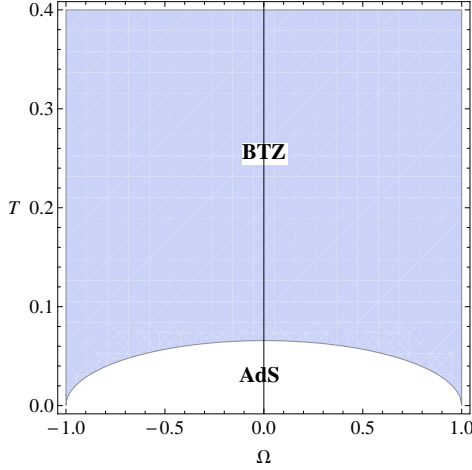


Figure 1: $m = 1.05$.

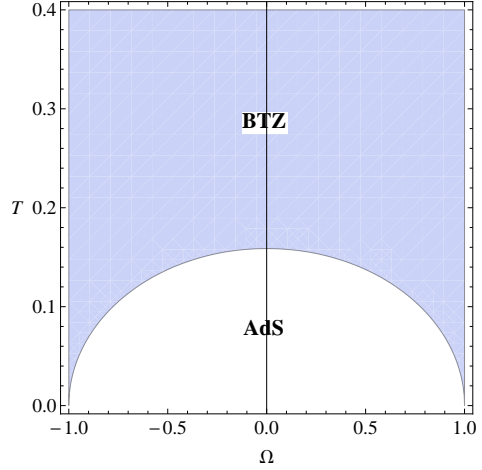


Figure 2: $m = 10$.

From the above diagrams, one can also notice that by decreasing m the effect of the higher derivative correction terms to the Einstein-Hilbert action would increase. This effect would make the BTZ black hole to form in a lower temperature. So forming a black hole in NMG is easier relative to the pure Einstein gravity due to the fact that in NMG the modes are massive. On the other hand, increasing m with a specific angular velocity would cause that the phase transition from AdS_3 to BTZ occurs at a bigger temperature.

5 Phase transitions of warped AdS_3 solution in quadratic ensemble

Now in this section we introduce the thermal WAdS_3 and Warped BTZ black hole solutions and then we present the phase diagrams.

Both of these solutions have an asymptotic warped AdS geometry which are the squashed or stretched deformation of AdS_3 with a different symmetry algebra than the AdS case. This dissimilarity of the algebra makes the thermal properties different from the asymptotic AdS solution as well. We derive some relations for deriving the thermal action of the warped solutions in NMG.

5.1 Gödel space-time

The time-like WAdS_3 or the three-dimensional Gödel space-time is the true vacuum of the WAdS_3 black hole [21] [30]. The original Gödel solution of the Einstein equations was four-dimensional. The non-trivial three-dimensional factor of Gödel space-time which is within the family of deformed AdS_3 were first studied in [3]. This metric is a constant curvature Lorentzian manifold with isometry group $U(1) \times SL(2, \mathbb{R})$ where the $U(1)$ factor is generated by a time-like Killing vector. As this metric can be embedded in the seven-dimensions flat space, it would possess time-like closed curves. However it is still a solution of string theory which corresponds to the time-like generator of $SL(2, \mathbb{R})$ or a magnetic background [6].

Its metric is given by

$$ds^2 = -dt^2 - 4\omega r dt d\phi + \frac{\ell^2 dr^2}{(2r^2(\omega^2 \ell^2 + 1) + 2\ell^2 r)} - \left(\frac{2r^2}{\ell^2} (\omega^2 \ell^2 - 1) - 2r \right) d\phi^2. \quad (5.1)$$

In the special case of $\omega^2 \ell^2 = 1$, this metric corresponds to AdS_3 .

For this timelike solution we would have

$$m^2 = -\frac{(19\omega^2 \ell^2 - 2)}{2\ell^2}, \quad \Lambda = -\frac{(11\omega^4 \ell^4 + 28\omega^2 \ell^2 - 4)}{2\ell^2(19\omega^2 \ell^2 - 2)}. \quad (5.2)$$

This metric were hugely studied in cosmological models although it contains CTSs and is unstable with respect to the quantum fluctuations. As these casual pathologies are large scale deficiencies, some rotating objects with physical applications in cosmology or perhaps in condensed matter could be modeled by this metric surrounded by a more standard space-time [3]. Therefore constructing the phase diagrams of this manifold could have interesting applications.

5.2 Space-like warped BTZ black hole

The warped AdS_3 or warped BTZ black hole in NMG in the quadratic non-local ensemble in its ADM form can be written as

$$\begin{aligned} \frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(\nu^2 + 3)(r - r_+)(r - r_-)} + (2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}) dt d\varphi \\ + \frac{r}{4} \left[3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu \sqrt{r_+ r_- (\nu^2 + 3)} \right] d\varphi^2. \end{aligned} \quad (5.3)$$

If $\nu^2 = 1$ the space is locally AdS_3 , if $\nu^2 > 1$ it is stretched and if $\nu^2 < 1$ it is a squashed deformation of AdS_3 .

For the space-like solution, the parameters would be,

$$m^2 = -\frac{(20\nu^2 - 3)}{2l^2}, \quad \Lambda = -\frac{m^2(4\nu^4 - 48\nu^2 + 9)}{(20\nu^2 - 3)^2}. \quad (5.4)$$

If one employs the following relations

$$\omega = \frac{\nu}{l}, \quad \omega^2 \ell^2 + 2 = 3\ell^2/l^2, \quad (5.5)$$

one reaches to equation 5.2 again.

Notice that “ l ” is the radius of space-like AdS_3 and “ ℓ ” is the radius of the warped time-like AdS_3 . Similar to the way that one can derive BTZ black hole by global identifications, one can also derive 5.3 from 5.1.

In order to have a real m and negative Λ and therefore a physical solution, from 5.2 and 5.4 the allowed range of ν and ω would be found as

$$-\sqrt{\frac{2}{19}} < \omega \ell < \sqrt{\frac{2}{19}}, \quad -\sqrt{\frac{3}{20}} < \nu < \sqrt{\frac{3}{20}}. \quad (5.6)$$

5.3 The free energies and phase diagrams

Now by using the thermodynamic quantities and conserved charges, we calculate the free energies of both of these space-times and then we proceed by making the phase diagrams.

Notice that the isometry group of the time-like WAdS₃ is $SL(2, \mathbb{R}) \times U(1)$ which is generated by four Killing vectors [21]. By assuming a specific boundary condition, the authors in [21] derived the asymptotic algebra of WAdS in NMG and then the central charge c and the $\hat{u}(1)_k$ kač-Moody level k as [31]

$$c = \frac{48\ell^4\omega^3}{G(19\ell^4\omega^4 + 17\ell^2\omega^2 - 2)} = -\frac{96l\nu^3}{G(20\nu^4 + 57\nu^2 - 9)}, \quad (5.7)$$

$$k = \frac{8\omega(1 + \ell^2\omega^2)}{G(19\ell^2\omega^2 - 2)} = \frac{4\nu(\nu^2 + 3)}{Gl(20\nu^2 - 3)}. \quad (5.8)$$

Now if we just simply assume that the relation 4.3 can be used here and the modular parameter is $\tau = \frac{1}{2\pi}(-\beta\Omega_E + i\frac{\beta}{l})$, then by using the above central charge one can find the free energy as

$$G_{\text{timelike WAdS}} = -\frac{4\ell^2\omega^3(\omega^2\ell^2 + 2)}{3G(19\ell^4\omega^4 + 17\ell^2\omega^2 - 2)} = -\frac{4\nu^2}{G(20\nu^2 - 3)} \times \frac{\nu}{(\nu^2 + 3)l}. \quad (5.9)$$

We can also recalculate the free energy by using the conserved charges. These conserved charges of the timelike WAdS₃ in NMG have been calculated in [31] for a “*spinning defect*”. Using these relations one can take the limit of $\mu \rightarrow 0$ to find the mass and angular momentum as

$$\mathcal{M} = -\frac{4\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)}, \quad \mathcal{J} = -\frac{4j\ell^4\omega^3}{G(19\ell^2\omega^2 - 2)}. \quad (5.10)$$

For the time-like warped AdS₃, again the entropy and the temperature are zero. So the Gibbs free energy, $G = \mathcal{M} - \Omega\mathcal{J}$, would be

$$G_{\text{spinning defect}} = \frac{4\ell^2\omega^2((\mu - 1) + \Omega j\ell^2\omega)}{G(19\ell^2\omega^2 - 2)}. \quad (5.11)$$

Taking the limit of zero defect, the result is as follows

$$G_{\text{timelike WAdS}} = -\frac{4\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)} = -\frac{4\nu^2}{G(20\nu^2 - 3)}. \quad (5.12)$$

Comparing 5.12 with 5.9, one can see that there is a factor of $N_1 = \frac{\nu}{(\nu^2 + 3)l}$ difference. This factor can be introduced in the modular parameter to compensate for this discrepancy.

For re-deriving this factor we can also calculate the free energy in a third way. As the authors in [31] found, the warped CFT version of the Cardy’s formula of entropy

$$S_{\text{WCFT}} = \frac{4\pi i}{k} \tilde{P}_0^{(\text{vac})} \tilde{P}_0 + 4\pi \sqrt{-\tilde{L}_0^{+(\text{vac})} \tilde{L}_0^+}, \quad (5.13)$$

matches with the black hole entropy

$$S_{BH} = \frac{8\pi\nu^3}{(20\nu^2 - 3)G_N} \left(r_+ - \frac{1}{2\nu} \sqrt{(\nu^2 + 3)r_- r_+} \right). \quad (5.14)$$

Now using the above equation and the relation 5.29 of the next section, one can find the warped BTZ

black hole entropy as

$$S_{\text{WBTZ}} = \frac{8\pi\nu^2}{G_N\Omega(20\nu^2 - 3)}. \quad (5.15)$$

If one uses the modular transformed equation for the BTZ black hole as

$$S = \frac{-i\pi c}{12l} \left(\frac{1}{\tau} - \frac{1}{\bar{\tau}} \right), \quad (5.16)$$

then by using the central charge 5.27, one can see that for matching the two relations, the modular parameter of the warped CFT should be defined as

$$\tau = \frac{2i\Omega\nu}{(\nu^2 + 3)}. \quad (5.17)$$

One can see that again a similar factor is appearing here. The imaginary factor here can point out to the appearance of closed time-like curves (CTCs) in the bulk.

This factor of $\frac{\nu}{(\nu^2+3)}$ can actually be explained by studying the Killing vectors of the space-time. The orbifold construction of warped AdS_3 preserves a $U(1) \times U(1)$ subgroup of $SL(2, \mathbb{R}) \times U(1)$ isometries and is generated by two Killing vectors

$$\xi^{(1)} = \partial_t, \quad \xi^{(2)} = \frac{2l\nu}{(\nu^2 + 3)} \partial_t + \partial_\varphi. \quad (5.18)$$

The same factor of $\frac{2l\nu}{(\nu^2+3)}$ is in the construction of the manifold which changes the partition function and therefore the free energy. This factor is the normalization factor $N_1 = \frac{2l\nu}{(\nu^2+3)}$ in [31] which is being fixed by matching the asymptotic Killing vector ℓ_0 with the vector $\xi^{(2)}$. In addition, in the WCFT context as in [11], this factor relates to the anomaly in the transformation operators T and P which generate the infinitesimal coordinate transformation in x and the gauge transformation in the gauge bundle, respectively.

Due to the current anomaly k , the operators T and P would mix with each other. This can be seen like a “tilt” (α) in the mapping from x^- to ϕ coordinates as in [11],

$$x_- = e^{i\phi}, \quad x^+ = t + 2\alpha\phi. \quad (5.19)$$

This spectral flow parameter α which is a property of the specific theory on the cylinder can be related to the factor N_1 for any theory. So in general, for the warped AdS_3 solutions one cannot simply use the relation 4.3. However for calculating the thermal action for space-times with warped AdS_3 asymptotes, one can redefine the modular parameter using the Killing vectors of the manifold or the normalization constant in the symmetry algebra of the asymptotic geometry.

This redefinition of the modular parameters have been also seen in Kerr/CFT contexts [32]. Specifically the NHEK geometry has an enhanced $SL(2, \mathbb{R}) \times U(1)$ isometry group where a different normalization factor appears in the algebra and in the normalization factors between the Killing vectors, and therefore in the redefinition of the modular parameter.

Now using the method introduced in previous chapters for calculating the conserved charges, we calculate the thermodynamic properties and the Gibbs free energies of the black holes with asymptotic warped AdS₃ geometry which could be called “warped BTZ black hole”. The thermodynamical quantities are [27], [21]

$$T_H = \frac{\nu^2 + 3}{8\pi\nu l} \left(\frac{r_+ - r_-}{r_+ - \sqrt{\frac{(\nu^2 + 3)r_+ r_-}{4\nu^2}}} \right), \quad \Omega_H = \frac{1}{\nu l} \left(\frac{1}{r_+ - \sqrt{\frac{(\nu^2 + 3)r_+ r_-}{4\nu^2}}} \right), \quad (5.20)$$

$$T_L = \frac{(\nu^2 + 3)}{8\pi l^2} (r_+ + r_- - \frac{1}{\nu} \sqrt{(\nu^2 + 3)r_- r_+}), \quad T_R = \frac{(\nu^2 + 3)}{8\pi l^2} (r_+ - r_-). \quad (5.21)$$

The conserved charges, mass and angular momentum are

$$\mathcal{M} = Q_{\partial_t} = \frac{\nu(\nu^2 + 3)}{Gl(20\nu^2 - 3)} \left((r_- + r_+)\nu - \sqrt{r_+ r_- (\nu^2 + 3)} \right), \quad (5.22)$$

$$\mathcal{J} = Q_{\partial_\varphi} = \frac{\nu(\nu^2 + 3)}{4Gl(20\nu^2 - 3)} \left((5\nu^2 + 3)r_+ r_- - 2\nu \sqrt{r_+ r_- (\nu^2 + 3)} (r_+ + r_-) \right). \quad (5.23)$$

Also the condition for the existence of black hole is

$$\mathcal{J} \leq \frac{Gl(20\nu^2 - 3)}{4\nu(\nu^2 + 3)} \mathcal{M}^2, \quad \mathcal{M} \geq 0, \quad (5.24)$$

which specifically does not put any new constraint on ν .

The entropy of warped BTZ black hole in NMG is again

$$S_{BH} = \frac{8\pi\nu^3}{(20\nu^2 - 3)G} \left(r_+ - \frac{1}{2\nu} \sqrt{(\nu^2 + 3)r_+ r_-} \right). \quad (5.25)$$

These thermodynamical quantities follow the first law of thermodynamics and their integrals follow the Smarr-like relation

$$M = T_H S_{BH} + 2\Omega_H J. \quad (5.26)$$

The central charge is [21]

$$c = -\frac{96l\nu^3}{G(20\nu^4 + 57\nu^2 - 9)}. \quad (5.27)$$

One can study the behavior of the central charge versus the warping parameter ν in Figure 4. In the region where there is no CTSs, it is a monotonically increasing function of ν . Also at $\nu = 0$ or $\nu \rightarrow \pm\infty$, one can see that the central charge is zero which indicates that for infinitely squashed or stretched space time the Casimir energy would be zero. The central charge also diverges at $\nu = \pm\sqrt{\frac{3}{20}} \sim 0.387$.

For having a physical theory the central charge should be positive. So if we assume that $G = l = 1$, then the constraint on ν would be $0 < \nu < \sqrt{\frac{3}{20}}$.

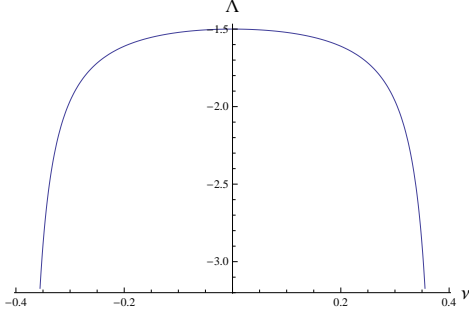


Figure 3: The plot of cosmological constant, Λ vs. $-\sqrt{\frac{3}{20}} < \nu < \sqrt{\frac{3}{20}}$.

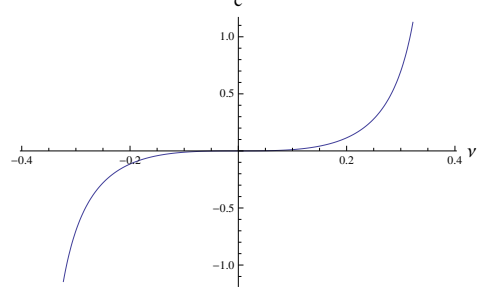


Figure 4: The central charge of NMG vs. ν .

So now if one defines

$$K = 1 + \frac{8l\pi T(l\pi T - \nu) \pm 4\pi l T \sqrt{4\pi^2 l^2 T^2 - 8l\pi T\nu + \nu^2 + 3}}{\nu^2 + 3}, \quad (5.28)$$

then

$$r_+ = \frac{1}{\Omega\nu l \left(1 - \frac{1}{2\nu} \sqrt{K(\nu^2 + 3)}\right)}, \quad r_- = Kr_+. \quad (5.29)$$

The minus sign in K would be acceptable which indeed could make r_- smaller than r_+ . Then the Gibbs free energy in terms of Ω, T, K and ν would be

$$G_{WBTZ} = \frac{1}{G\Omega l(20\nu^2 - 3)} \left[-8\pi T\nu^2 + \frac{(\nu^2 + 3)}{l \left(1 - \frac{1}{2\nu} \sqrt{K(\nu^2 + 3)}\right)} \left((K+1)\nu - \sqrt{K(\nu^2 + 3)} \right) - \frac{(5\nu^2 + 3)K - 2\nu(K+1)\sqrt{K(\nu^2 + 3)}}{4\nu l \left(1 - \frac{1}{2\nu} \sqrt{K(\nu^2 + 3)}\right)} \right].$$

Notice that this Gibbs free energy only depends on Ω, T and ν .

One should also notice that the limit of the un-warped black hole is $\nu = \beta^2 = 1$ which corresponds to $m = 1$ for $l = 1$. In this limit the G_{WBTZ} does not reach to the limit of G_{BTZ} in equation 4.23. This is specially obvious from the factors of Ω and T . Actually this is not a real problem as we shouldn't expect to reach to the results of the previous section for the case of $\nu \rightarrow 1$ since these metrics have been written in two different coordinate systems.

Now that we have found the free energies of the warped BTZ black hole and its vacuum, we can find the phase diagrams of temperature versus the angular velocity as before and then we can compare them for different warping factors. Therefore we can study the effect of ν on the phase transitions in warped geometries.

We saw that the acceptable interval for ν is $0 < \nu < \sqrt{\frac{3}{20}} \sim 0.3872$. The phase diagram for $\nu = 0.387$ is shown in Figure 5. The blue regions are where the warped BTZ black hole is the dominant phase and in the white regions the vacuum WAdS is dominant. If one increases ν till $\nu \geq \sqrt{\frac{3}{20}}$, the places of these

two phases change with each other as it is also obvious from the functionality of the central charge in terms of the warping factor ν .

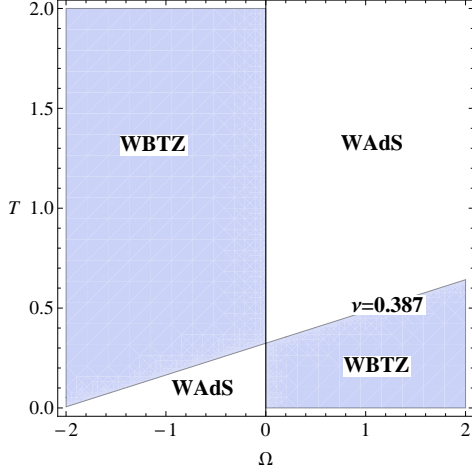


Figure 5: The phase diagram for $\nu = 0.387$.

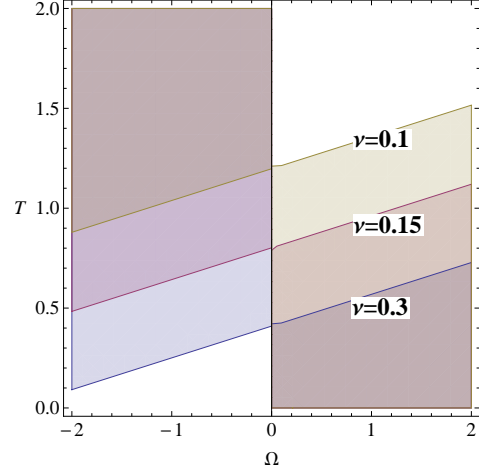


Figure 6: The phase diagram for different ν .

In these diagrams one can notice some other points. As one can see in the figures and also from the equations of the free energy of warped BTZ black hole, in the quadratic/non-local ensemble, unlike the grand canonical ensemble, the diagrams of warped AdS solutions are not symmetric although the NMG is parity preserving. One can notice that the behaviors for the positive and negative angular velocities are very different as the free energy is an odd function of Ω for the warped geometry unlike the previous case.

Also, if $\Omega > 0$ the phase of warped black hole is dominant at lower temperatures and the thermal warped AdS₃ is dominant at higher temperatures. However, in this case a higher temperature would trigger the reverse Hawking-Page transition and the black hole phase changes to the thermal warped AdS₃. So bigger Ω makes the black hole the dominant phase while bigger T makes the vacuum dominant.

Furthermore, the effect of warping factor ν is shown in Figure 6. If we define a critical temperature T_c where the tilted line crosses the $\Omega = 0$ axis, then one can see that increasing ν would decrease this critical temperature. So for $\Omega < 0$, increasing the warping factor ν makes the black hole phase more dominant and for $\Omega > 0$, it makes the vacuum AdS₃ the dominant phase.

6 Phase diagram of warped AdS₃ solution in grand canonical ensemble

The WAdS black hole in the grand canonical solution would be of the following form

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r^2}{l^2} - \frac{H^2(-r^2-4lJ+8l^2M)^2}{4l^3(lM-J)} + 8M & 0 & 4J - \frac{H^2(4lJ-r^2)(-r^2-4lJ+8l^2M)}{4l^2(lM-J)} \\ 0 & \frac{1}{\frac{16J^2}{r^2} + \frac{r^2}{l^2} - 8M} & 0 \\ 4J - \frac{H^2(4lJ-r^2)(-r^2-4lJ+8l^2M)}{4l^2(lM-J)} & 0 & r^2 - \frac{H^2(4lJ-r^2)^2}{4l(lM-J)} \end{pmatrix}. \quad (6.1)$$

The change of coordinates to go from this form to the form of 5.3 were derived in [2]. Also the phase diagram of this specific ensemble was just recently presented in [33]. Here we brought the phase diagram

of this ensemble for the sake of comparison to the previous case.

Now using the charges and entropy derived in 3.26 and 3.27, one can derive the Gibbs free energy as

$$G_{WBTZ}(T, \Omega) = \frac{-8l^2\pi^2 T^2(1-2H^2)^{\frac{3}{2}}}{(17-42H^2)(1-l^2\Omega^2)}, \quad (6.2)$$

and the vacuum corresponds to $M = -\frac{1}{8}$ and $J = 0$.

6.1 local stability

The Hessian matrix would be

$$H = \begin{pmatrix} \frac{16l^2\pi^2(1-2H^2)^{3/2}}{(17-42H^2)(l^2\Omega^2-1)} & \frac{-32l^4\pi^2 T\Omega(1-2H^2)^{3/2}}{(17-42H^2)(l^2\Omega^2-1)^2} \\ \frac{-32l^4\pi^2 T\Omega(1-2H^2)^{3/2}}{(17-42H^2)(l^2\Omega^2-1)^2} & \frac{16\pi^2 T^2(1-2H^2)^{3/2}(l^4+3l^6\Omega^2)}{(17-42H^2)(l^2\Omega^2-1)^3} \end{pmatrix}. \quad (6.3)$$

For having a locally stable solution, both of the eigenvalues of the Hessian should be negative. For the case of $\Omega = 0$ the condition of making both eigenvalues negative would be $H^2 < \frac{17}{42}$.

One can notice that unlike the previous ensemble, in the grand canonical ensemble the diagrams of warped BTZ black hole solution in addition to BTZ black holes would be symmetric. This could be just the result of the symmetry and parity preserving nature of this kind of solutions in BHT gravity.

Also this could show us that the thermodynamical properties of these black holes and therefore the Hawking-Page phase diagrams could only really be meaningful in the grand canonical ensemble and not in other ensembles. The significations and the dual interpretations of the phase diagrams in other thermodynamical ensembles is not particularly clear.

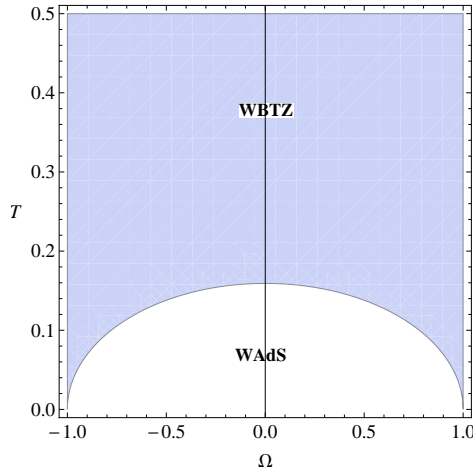


Figure 7: Phase diagram for WAdS solution in grand canonical ensemble, $C = l = 1$.

7 Phase diagram of the hairy black hole

There exist yet another interesting black hole solution in the new massive gravity, the “New Hairy black hole”. In this section, we are interested in studying its Hawking-Page phase transitions as well.

This solution was first introduced in [34] and later it was studied more in [27], [35] and [36]. Its geodesics and entanglement entropy for the specific case of non-rotating solution were discussed recently in [37].

This hairy black hole solution exists for $m^2 = \Lambda = -\frac{1}{2l^2}$ for the parameters of action 2.1. The form of its metric is as follows

$$ds^2 = -NFdt^2 + \frac{dr^2}{F} + r^2(d\phi + N^\phi dt)^2, \quad (7.1)$$

where

$$\begin{aligned} N &= \left(1 + \frac{bl^2}{4H} \left(1 - \Xi^{\frac{1}{2}}\right)\right)^2, & N^\phi &= -\frac{a}{2r^2} (4G_N M - bH), \\ F &= \frac{H^2}{r^2} \left(\frac{H^2}{l^2} + \frac{b}{2} \left(1 + \Xi^{\frac{1}{2}}\right) H + \frac{b^2 l^2}{16} \left(1 - \Xi^{\frac{1}{2}}\right)^2 - 4G_N M \Xi^{\frac{1}{2}}\right), \\ H &= \left(r^2 - 2G_N M l^2 \left(1 - \Xi^{\frac{1}{2}}\right) - \frac{b^2 l^4}{16} \left(1 - \Xi^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}, \end{aligned}$$

and the definition of the parameter is $\Xi := 1 - \frac{a^2}{l^2}$. Also $-l \leq a \leq l$. Now there are two conserved charges for this black hole which are M and $J = Ma$ and also a gravitational hair parameter which is b .

The thermodynamical parameters of this black hole are

$$\begin{aligned} \Omega_+ &= \frac{1}{a} \left(\Xi^{\frac{1}{2}} - 1\right), & T &= \frac{1}{\pi l} \Xi^{\frac{1}{2}} \sqrt{2G_N \Delta M \left(1 + \Xi^{\frac{1}{2}}\right)^{-1}}, \\ S &= \pi l \sqrt{\frac{2}{G_N} \Delta M \left(1 + \Xi^{\frac{1}{2}}\right)}, & \Delta M &= M + \frac{b^2 l^2}{16G_N}. \end{aligned}$$

Then using all of these thermodynamical quantities one can read the Gibbs free energy. We will see that the region where the black hole can be locally stable for any b is $\Omega^2 l^2 < 1$. So with this assumption we can simplify the relation as

$$G_{NBH} = \frac{l^2}{16G} \left(\frac{16\pi^2 T^2 (5l^2 \Omega^2 - 1)}{(l^2 \Omega^2 - 1)^2} - \frac{b^2 (3l^2 \Omega^2 + 1)}{l^2 \Omega^2 + 1} \right). \quad (7.2)$$

We can see that the Gibbs free energy, in addition to Ω and T , depends also on the hair parameter b . One can also notice that there is no real b which makes this free energy vanish.

Now for studying the local stability we calculate the Hessian as

$$H = \begin{pmatrix} \frac{2l^4 \pi^2 \Omega^2 (l^2 \Omega^2 + 3)}{G(l^2 \Omega^2 - 1)^2} & -\frac{4l^4 \pi^2 T \Omega (5l^2 \Omega^2 + 3)}{G(l^2 \Omega^2 - 1)^3} \\ -\frac{4l^4 \pi^2 T \Omega (5l^2 \Omega^2 + 3)}{G(l^2 \Omega^2 - 1)^3} & \frac{l^4 (b^2 (l^2 \Omega^2 - 1)^4 (3l^2 \Omega^2 - 1) + 24\pi^2 T^2 (l^2 \Omega^2 + 1)^3 (1 + 5l^2 \Omega^2 (2 + l^2 \Omega^2)))}{4G(l^2 \Omega^2 - 1)^4 (l^2 \Omega^2 + 1)^3} \end{pmatrix}. \quad (7.3)$$

The region where both of the eigenvalues of the above matrix is negative would depend on the hair parameter b . The phase diagram for a specific value of $b = 20$ is shown in figure 8.

For $G_N = l = 1$ one can check that for any b the angular velocity should be in the range of $-0.5 < \Omega < 0.5$, so that the black hole solution can be locally stable. Increasing Ω can make the black hole locally unstable. Also the condition for T depends on b .

Increasing the hair parameter b would make the locally stable region bigger. So basically the hair parameter makes the system more stable and Ω makes it more unstable. In condensed matter systems, one can also investigate the dual interpretation of this hair parameter and show how it makes the system stable.

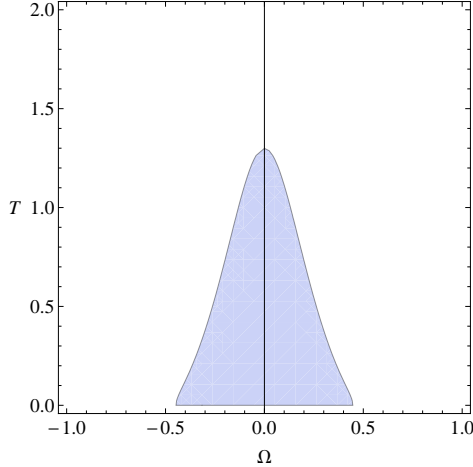


Figure 8: The local stable region for $b = 20$.

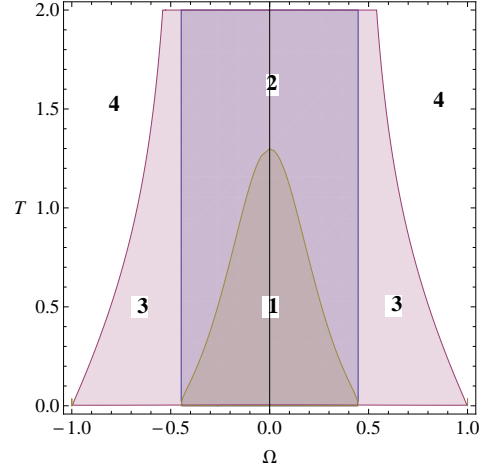


Figure 9: The phase diagram for $b = 20$.

We now can compare the Gibbs free energies of this black hole with the free energies of other solutions to study the phase diagram of the system. The phase diagram for the region of local stability, the vacuum AdS_3 solution and the ground state of NBH is shown in figure 9. The region where $\Delta G_1 = G_{\text{AdS}} - G_{\text{NBH}} > 0$ is the union of regions 1, 2, 3. By comparing the Gibbs free energy of this black hole with the the free energy of the vacuum AdS, one could see that for the region of locally stable, for any b only the black hole phase would be present. So the only phase that is both locally and globally stable is the black hole case. Outside of region 1, the phase would not be even locally stable.

Also the case of $M = M_0 = -\frac{b^2 l^2}{16G}$ in this solution is the ground state which corresponds to an extremal case where both the left, right and the Hawking temperature and also the entropy vanish. [35]. So its free energy would be

$$G_{0\text{NBH}} = -\frac{b^2 l^2}{16G_N} \left(3 - \frac{2}{1 + l^2 \Omega^2} \right). \quad (7.4)$$

The region where $\Delta G_2 = G_{0\text{NBH}} - G_{\text{NBH}} > 0$ is the union of 1 and 2. Again one can see that $-0.5 < \Omega < 0.5$ is the region where the black hole can be stable.

One can also plot the diagram of M versus J and study the effect of other physical parameters or conserved charges on the phase transitions which could shed light on other physical characteristics.

8 The inner horizon thermodynamics

It would also be useful to study the thermodynamics of black holes *inside* the horizon as the quantities from outside of the horizon combined with the inside ones can provide additional information about the central charges or scattering data around the black hole. Also the relations between the thermodynamics of inside and outside of horizon would be of practical use in holographical applications such as the examples in [2], [38].

One can first integrate the Wald's formula on $r = r_-$ to find the inner horizon entropy. Alternatively, one can use the following relations,

$$T_{R,L} = \frac{T_- \pm T_+}{\Omega_- - \Omega_+}, \quad S_{\pm} = S_R \pm S_L, \quad S_{R,L} = \frac{\pi^2 l}{3} c T_{R,L}. \quad (8.1)$$

So one would get,

$$S_- = \frac{8\pi\nu^3}{(20\nu^2 - 3)G} (r_- - \frac{1}{2\nu} \sqrt{(\nu^2 + 3)r_+ r_-}). \quad (8.2)$$

The temperature at the inner horizon would be

$$T_- = \frac{1}{2\pi l} \frac{\partial_r N}{\sqrt{g_{rr}}} \Big|_{r=r_-} = \frac{\nu^2 + 3}{8\pi\nu} \left(\frac{r_+ - r_-}{r_- - \sqrt{\frac{(\nu^2 + 3)r_+ r_-}{4\nu^2}}} \right), \quad (8.3)$$

and the angular velocity at the inner horizon is

$$\Omega_- = \frac{1}{l} N^\phi(r_-) = \frac{1}{\nu l} \left(\frac{1}{r_- - \sqrt{\frac{(\nu^2 + 3)r_+ r_-}{4\nu^2}}} \right). \quad (8.4)$$

As explained in [38], the statement of inner horizons mechanics is that the product of all horizons' entropies should be proportional to the conserved charges that at the quantum level are quantized. In this case this charge is only J . The product of the inner and outer horizon entropies would be

$$S_+ S_- = \frac{16\pi^2 \nu^4}{G^2 (20\nu^2 - 3)^2} \left((5\nu^2 + 3)r_+ r_- - 2\nu \sqrt{(\nu^2 + 3)r_+ r_-} (r_+ + r_-) \right), \quad (8.5)$$

which as expected can be written as a factor of J , i.e, $S_+ S_- \propto J$, and it is independent of M . This is because $S_+ S_-$ is holography dual to the level matching condition of the warped CFT₂ and this is also consistent with the WAdS/WCFT picture.

Also as explained in [39], the mass-independency of the product of entropies would be satisfied here as $T_+ S_+ = T_- S_-$ which is also a result of the parity-preserving nature of this theory, i.e., the fact that the left and right central charges are equal. Also based on [40], as the Smarr-relation holds for the Einstein gravity with the higher curvature corrections in the form of NMG, we would expect such result. Moreover, the first law of black hole thermodynamics would be satisfied as

$$dM = \pm T_{\pm} dS_{\pm} + \Omega_{\pm} dJ. \quad (8.6)$$

This was expected since as explained in [39], if the first law is satisfied in outer horizon, it should also be satisfied in the inner-horizon as well. Then using both of these relations, one can derive the same results for the M and J as the equations 5.22 and 5.23, consistent with WAdS/WCFT picture [41].

9 Entanglement entropy of WCFT in BHT

Recently using the Rindler method, the authors of [13] found a general expression for the entanglement and the Renyi entropy of a single interval in a $(1+1)$ -dimensional WCFT. Then they have provided the geodesic and massive point particle description in a lower spin gravity as the dual of a WCFT. This way they also could holographically study the results from the bulk geometry.

Their general result of the entanglement entropy of an interval in the warped CFT theories is as follows

$$S_{EE} = iP_0^{\text{vac}} \ell^* \left(\frac{\bar{L}}{L} - \frac{\bar{\ell}^*}{\ell^*} \right) - 4L_0^{\text{vac}} \log \left(\frac{L}{\pi\epsilon} \sin \frac{\pi\ell^*}{L} \right). \quad (9.1)$$

In the above formula ℓ^* and $\bar{\ell}^*$ are the separation in space and time respectively and L and \bar{L} are related to the identification pattern of the circle that defines the vacuum of the theory [13].

The authors have explained that the second term is well-known and expected, but the first term seems exotic. With the motivation of finding the nature of the first term, we study this result for our specific case of NMG theory.

Since the theory is parity even, for the vacuum of WAdS₃ which is holographically dual to WCFT one can write the Virasoro and Kac-Moody operators as

$$\tilde{P}_0^{(\text{vac})} = \mathcal{M}^{(\text{vac})}, \quad \tilde{L}_0^{(\text{vac})} = \frac{1}{k} (\mathcal{M}^{(\text{vac})})^2, \quad (9.2)$$

where as found in [21], the mass parameter is

$$\mathcal{M}^{(\text{vac})} = i\mathcal{M}_{\text{God}} = -i \frac{4\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)}. \quad (9.3)$$

Here, as expected, $P_0^{(\text{vac})}$ has an overall factor of i which makes the first term of 9.1 real.

It would also be interesting to write the Virasoro and U(1) Kac-Moody operators of NMG in terms of its central charge as well. So using 9.2 and 9.3 and the expression for c and k in eq. 5.7, one can write

$$\tilde{P}_0^{(\text{vac})} = -\frac{ic}{12l^2\omega}(1 + \ell^2\omega^2), \quad \tilde{L}_0^{(\text{vac})} = -\frac{c}{24}. \quad (9.4)$$

We can also compare it with the operators of TMG,

$$\tilde{P}_0^{(\text{vac})} = -\frac{c_L}{24}, \quad \tilde{L}_0^{(\text{vac})} = -\frac{c_R}{24}. \quad (9.5)$$

One can see that the Virasoro part is only proportional to c_R , similar to the TMG case. However, the Kac-Moody part does not only depend on c but also on ω , and therefore it cannot define a central charge. Still c_L is called a central charge only by convention. These relations could be useful for studying the dual WCFT of time-like Gödel or warped BTZ in these massive gravitational theories.

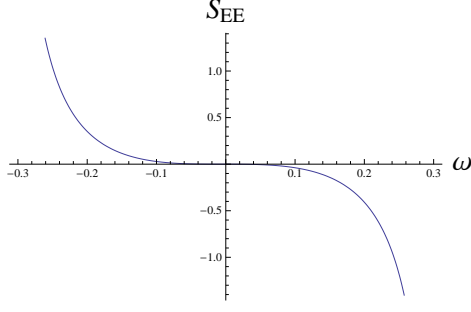


Figure 10: The plot of S_{EE} versus ω .

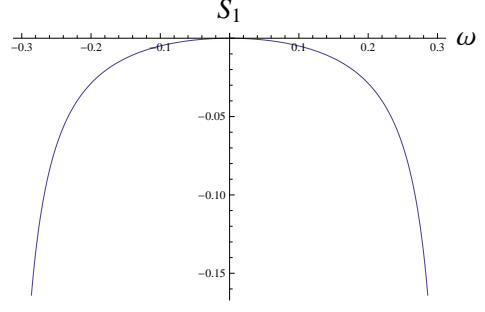


Figure 11: The plot of S_1 versus ω .

Now by using 9.2, the entanglement entropy of a strip in a WCFT theory dual to a NMG bulk could be found as

$$S_{EE} = \frac{4\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)} \left(\ell^* \left(\frac{\bar{L}}{L} - \frac{\bar{\ell}^*}{\ell^*} \right) + \frac{2\ell^2\omega}{1 + \ell^2\omega^2} \log \left(\frac{L}{\pi\epsilon} \sin \frac{\pi\ell^*}{L} \right) \right). \quad (9.6)$$

Notice that ℓ here is just the length of AdS radius in the Gödel space-time and is independent of the length of intervals ℓ^* and $\bar{\ell}^*$.

The plot of S_{EE} versus ω is shown in Figure 10. The lengths are considered to be constant and equal to one, so one can examine the effect of the warping factor of Gödel space-time on the entanglement entropy. As it is shown, the entanglement entropy is a monotonic function of ω . One can also see that at $\omega = \pm\sqrt{\frac{2}{19}}$ the entanglement entropy diverges.

Also the plot of the peculiar first term contributing to the entanglement entropy for NMG versus the parameter ω is shown in Fig. 11 where its relation is

$$S_1 = \frac{\omega k}{2(1 + \ell^2\omega^2)} \left(\frac{\bar{L}}{L} - \frac{\bar{\ell}^*}{\ell^*} \right) \ell^2 \ell^*. \quad (9.7)$$

One can notice that, based on the sign of $\left(\frac{\bar{L}}{L} - \frac{\bar{\ell}^*}{\ell^*} \right)$, it can be a positive or negative term and has an extremum at $\omega = 0$ and also it is a symmetric function.

One can also write ω and ℓ in terms of the physical quantities m and Λ as

$$\omega = \pm \sqrt{\frac{-2m^2}{7} \pm \frac{\sqrt{5m^4 - 7m^2\Lambda}}{7\sqrt{3}}}, \quad \ell = \pm \sqrt{\frac{144m^2 \pm 38\sqrt{3m^2(5m^2 - 7\Lambda)}}{11m^4 - 361m^2\Lambda}}. \quad (9.8)$$

Using the above relations one can study the relationships between the entanglement entropy and physical parameters m and Λ as well.

As explained in [13], in the holographic picture, the first term contributes to the black hole entropy. This term is a U(1) contribution to the entanglement entropy and is not UV divergent like the second term. It is also proportional to the volume of the interval although it corresponds to the vacuum states and not the mixed states which is a new observation in warped AdS geometries. Also as noted in [13], this term in WCFT, unlike CFT, is independent of the Renyi replica index. It is also a rather complicated function of physical parameter of the theory such as m and Λ as in relations 9.8.

10 Discussion

In this paper we calculated the free energies of the thermal AdS_3 and BTZ black holes, thermal warped AdS_3 and warped BTZ black holes in the quadratic/non-local and grand canonical ensembles and also the new hairy black hole in BHT gravity and then we plotted the Hawking-Page phase transition diagrams. We found symmetric diagrams for the solutions with AdS_3 geometry, warped AdS geometry in grand canonical ensemble and non-symmetric diagrams for the solution with warped AdS_3 geometry in quadratic ensemble. Since the theory of BHT is parity preserving, this could lead to the fact that only the grand canonical ensemble could be used to present the physical phase diagram. By using the phase diagrams, we also studied the effect of mass parameters, warping factor ν and the hair parameter.

For calculating the free energy of the vacuum warped AdS_3 , we should change the previous relations for the action of thermal AdS_3 to the deformed case. By calculating the free energy from the conserved charges or the Cardy-like formula for the warped solutions in NMG, we found a factor, $\frac{\nu}{(\nu^2+3)l}$ that should multiply to one of the terms in the definition of the modular parameter of the torus for compensating the difference in the result of different methods. This factor extends the definition of the modular parameter of the torus and the formula for the action of thermal AdS_3 to the warped solutions in NMG theory.

Also the thermodynamics of the inner horizon have been studied which were consistent with the previous results and the WAdS/WCFT picture. We found that the product of the entropies of the inner and outer horizon is a product of J and therefore independent of mass. This result verifies the inner horizon universality property for the warped black hole solution of the new massive gravity as well.

In the last section, using the general formula of [13], we found the entanglement entropy of an interval in a WCFT dual to a warped AdS_3 geometry in NMG theory and then we studied the behaviors of the two terms contributing to the entanglement entropy versus the warping factor. Also one could try to derive the same result using the HRT formalism of deriving the entanglement entropy holographically [37], [42] or study the geodesics in time-like warped AdS_3 (Gödel space-time) or space time WAdS₃, which could be done in the future works.

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References

- [1] A. Adams, D. A. Roberts and O. Saremi, *Hawking-Page transition in holographic massive gravity*, *Phys. Rev.* **D91** (2015) 046003, [1408.6560].
- [2] S. Detournay and C. Zwikel, *Phase transitions in warped AdS_3 gravity*, *JHEP* **05** (2015) 074, [1504.00827].
- [3] M. Rooman and P. Spindel, *Gödel metric as a squashed anti-de Sitter geometry*, *Class. Quant. Grav.* **15** (1998) 3241–3249, [gr-qc/9804027].

- [4] D. Israel, *Quantization of heterotic strings in a Godel / anti-de Sitter space-time and chronology protection*, *JHEP* **01** (2004) 042, [[hep-th/0310158](#)].
- [5] S. Detournay, D. Israel, J. M. Lapan and M. Romo, *String Theory on Warped AdS_3 and Virasoro Resonances*, *JHEP* **01** (2011) 030, [[1007.2781](#)].
- [6] D. Israel, C. Kounnas, D. Orlando and P. M. Petropoulos, *Electric/magnetic deformations of S^{**3} and $AdS(3)$, and geometric cosets*, *Fortsch. Phys.* **53** (2005) 73–104, [[hep-th/0405213](#)].
- [7] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, *Warped $AdS(3)$ Black Holes*, *JHEP* **03** (2009) 130, [[0807.3040](#)].
- [8] T. Azeyanagi, D. M. Hofman, W. Song and A. Strominger, *The Spectrum of Strings on Warped $AdS_3 \times S^3$* , *JHEP* **04** (2013) 078, [[1207.5050](#)].
- [9] W. Song and A. Strominger, *Warped AdS_3 /Dipole-CFT Duality*, *JHEP* **05** (2012) 120, [[1109.0544](#)].
- [10] M. J. Reboucas and J. Tiomno, *On the Homogeneity of Riemannian Space-Times of Godel Type*, *Phys. Rev.* **D28** (1983) 1251–1264.
- [11] S. Detournay, T. Hartman and D. M. Hofman, *Warped Conformal Field Theory*, *Phys. Rev.* **D86** (2012) 124018, [[1210.0539](#)].
- [12] D. Anninos, J. Samani and E. Shaghoulian, *Warped Entanglement Entropy*, *JHEP* **02** (2014) 118, [[1309.2579](#)].
- [13] A. Castro, D. M. Hofman and N. Iqbal, *Entanglement Entropy in Warped Conformal Field Theories*, [1511.00707](#).
- [14] S. W. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.
- [15] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034, [[hep-th/0506176](#)].
- [16] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Massive Gravity in Three Dimensions*, *Phys. Rev. Lett.* **102** (2009) 201301, [[0901.1766](#)].
- [17] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, *Lifshitz Black Hole in Three Dimensions*, *Phys. Rev.* **D80** (2009) 104029, [[0909.1347](#)].
- [18] S.-J. Zhang, *HawkingPage phase transition in new massive gravity*, *Phys. Lett.* **B747** (2015) 158–163.
- [19] S. Nam, J.-D. Park and S.-H. Yi, *Mass and Angular momentum of Black Holes in New Massive Gravity*, *Phys. Rev.* **D82** (2010) 124049, [[1009.1962](#)].
- [20] G. Clement, *Warped $AdS(3)$ black holes in new massive gravity*, *Class. Quant. Grav.* **26** (2009) 105015, [[0902.4634](#)].

- [21] L. Donnay and G. Giribet, *Holographic entropy of Warped-AdS₃ black holes*, *JHEP* **06** (2015) 099, [1504.05640].
- [22] F. Correa, M. Hassaine and J. Oliva, *Black holes in New Massive Gravity dressed by a (non)minimally coupled scalar field*, *Phys. Rev.* **D89** (2014) 124005, [1403.6479].
- [23] D. Grumiller and O. Hohm, *AdS(3)/LCFT(2): Correlators in New Massive Gravity*, *Phys. Lett.* **B686** (2010) 264–267, [0911.4274].
- [24] P. Kraus, *Lectures on black holes and the AdS(3) / CFT(2) correspondence*, *Lect. Notes Phys.* **755** (2008) 193–247, [hep-th/0609074].
- [25] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *More on Massive 3D Gravity*, *Phys. Rev.* **D79** (2009) 124042, [0905.1259].
- [26] Y. Liu and Y.-w. Sun, *Note on New Massive Gravity in AdS(3)*, *JHEP* **04** (2009) 106, [0903.0536].
- [27] S. Nam, J.-D. Park and S.-H. Yi, *AdS Black Hole Solutions in the Extended New Massive Gravity*, *JHEP* **07** (2010) 058, [1005.1619].
- [28] J. M. Maldacena and A. Strominger, *AdS(3) black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005, [hep-th/9804085].
- [29] Y. S. Myung, *Phase transitions of the BTZ black hole in new massive gravity*, *Adv. High Energy Phys.* **2015** (2015) 478273, [1510.02853].
- [30] M. Banados, G. Barnich, G. Compere and A. Gomberoff, *Three dimensional origin of Godel spacetimes and black holes*, *Phys. Rev.* **D73** (2006) 044006, [hep-th/0512105].
- [31] L. Donnay, J. J. Fernandez-Melgarejo, G. Giribet, A. Goya and E. Lavia, *Conserved charges in timelike warped AdS₃ spaces*, *Phys. Rev.* **D91** (2015) 125006, [1504.05212].
- [32] M. Guica, T. Hartman, W. Song and A. Strominger, *The Kerr/CFT Correspondence*, *Phys. Rev.* **D80** (2009) 124008, [0809.4266].
- [33] S. Detournay, L.-A. Douchamps, G. S. Ng and C. Zwickel, *Warped AdS₃ Black Holes in Higher Derivative Gravity Theories*, 1602.09089.
- [34] J. Oliva, D. Tempo and R. Troncoso, *Three-dimensional black holes, gravitational solitons, kinks and wormholes for BHT massive gravity*, *JHEP* **07** (2009) 011, [0905.1545].
- [35] G. Giribet, J. Oliva, D. Tempo and R. Troncoso, *Microscopic entropy of the three-dimensional rotating black hole of BHT massive gravity*, *Phys. Rev.* **D80** (2009) 124046, [0909.2564].
- [36] B. Mirza and Z. Sherkatghanad, *Phase transitions of hairy black holes in massive gravity and thermodynamic behavior of charged AdS black holes in an extended phase space*, *Phys. Rev.* **D90** (2014) 084006, [1409.6839].

- [37] S. M. Hosseini and . Vliz-Osorio, *Free-kick condition for entanglement entropy in higher curvature gravity*, *Phys. Rev.* **D92** (2015) 046010, [1505.00826].
- [38] G. Giribet and M. Tsoukalas, *Warped-AdS3 black holes with scalar halo*, *Phys. Rev.* **D92** (2015) 064027, [1506.05336].
- [39] B. Chen, S.-x. Liu and J.-j. Zhang, *Thermodynamics of Black Hole Horizons and Kerr/CFT Correspondence*, *JHEP* **11** (2012) 017, [1206.2015].
- [40] A. Castro, N. Dehmami, G. Giribet and D. Kastor, *On the Universality of Inner Black Hole Mechanics and Higher Curvature Gravity*, *JHEP* **07** (2013) 164, [1304.1696].
- [41] L. Donnay and G. Giribet, *WAdS₃/CFT₂ correspondence in presence of bulk massive gravitons*, in *14th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG14) Rome, Italy, July 12-18, 2015*, 2015. 1511.02144.
- [42] V. E. Hubeny, M. Rangamani and T. Takayanagi, *A Covariant holographic entanglement entropy proposal*, *JHEP* **07** (2007) 062, [0705.0016].